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On a New Class of Impulsive η-Hilfer Fractional Volterra-Fredholm Integro-Differential Equations

Ismaael, F. M.*[1]

[1]*Department of Mathematics, Open Education College, Ministry of Education, Iraq*

E-mail: fawzimuttar62@gmail.com
**Corresponding author*

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Abstract

This work addresses the idea of the uniqueness and existence results for a class of boundary value problems (BVPs) for implicit Volterra-Fredholm integro-differential equations (V-FIDEs) with fractional η-Hilfer nonlinear equations and multi-point fractional boundary non-instantaneous conditions. The conclusions are confirmed by the fixed point of Krasnoselskii's theorem and the Banach contraction principle. Finally, a concrete example is given to illustrate our main conclusions.

Keywords: fractional η-Hilfer derivative; V-FIDE; problem with boundaries; fixed point method.

1 Introduction

A 1695 correspondence between Guilliaume L'Hopital and Gottfried Leibniz over the potential relevance and meaning of non-integer-order derivatives served as the initial impetus for the idea of fractional calculus [19]. Thanks to the combined efforts of various mathematicians, notably Riemann, Grunwald, Letnikov, and Liouville, a fairly sound theory of fractional calculus for functions of a real variable had been constructed by the late nineteenth century [21]. Despite the publication of various probable fractional derivatives, the so-called Caputo and Riemann-Liouville (R-L) are currently the two most popular variations, to mention [11]. The fractional calculus theory has recently become an interesting topic of research for mathematicians, scientists, and engineers due to the appearance of this theory in several applications in natural science and engineering and the ability to model many systems and phenomena that have memory effects. For more resources about this theory, we suggest [7, 8].

Mathematicians have recently modeled and addressed a broad variety of applied problems using this fractional calculus. This is correct, as Podlubney states in [32]. In their studies on fractional differential equations, several writers in the literature concentrated on Caputo and R-L type derivatives [9]. Hilfer created the Hilfer fractional derivative of order α and a type β in $[0, 1]$, which interpolates between the R-L and Caputo derivatives, as a generalisation of both R-L and Caputo derivatives [29].

This justifies the usage of and generalization of the fractional integro-differential equations, Existence, uniqueness and stability [16], Uniqueness and stability results for Caputo type [12], existence results on Hadamard neutral [15], controllability [14], On time scales [13], existence and uniqueness [11], uniqueness [20], existence and stability results [34], β-Ulam-Hyers stability [42], existence and stability results for ψ-Hilfer [41], existence results for Hilfer [40], existence results of Katugampola type [25], existence and stability of solutions of ψ-Hilfer [22], existence and stability of $\Re$-Hilfer [17]. Using Hilfer operators, several scholars have recently looked into the uniqueness, existence, stability, and generalization of different BVPs [26].

Asawasamrit et al. in [6] investigated non-instantaneous impulsive BVPs and the η-Caputo (or, more accurately, η-Caputo-Liouville) fractional derivative. Ivaz et al. [23] investigated the η-Hilfer fractional derivative integrating boundary conditions. Authors in [4] produced new existence discoveries for FIDEs with impulsive and integral criteria. Ali et al. used the HOBW method to solve the fractional V-FIDEs with mixed boundary conditions in [3]. Agarwal et al. examined non-instantaneous impulses in Caputo FDEs in [2]. Hussain et al. in [20] studied fractional BVP with η-Caputo derivative. According to Kailasavalli et al.'s deduction in [24], there must be solutions for FIDEs in Banach spaces. In [30], Nuchpong et al. considered inclusions with Hilfer-type FIDE of BVPs and integro-multipoint nonlocal boundary conditions. Some essential FDE theory and applications are provided by Kilbas et al. Authors in [25] investigated existence findings for fractional impulsive IDEs using integral conditions of the Katugampola type.

In [32], Podlubny examined a few FDEs. Srivastava reviews the latest newly developed trends in fractional integrals and derivatives in [38]. In [37], Srivastava covered a few parametric and argument forms of the fractional calculus operators, along with related ideas such integral transformations and special functions. Srivastava in [39] presents an introduction to the Fox-Wright-based operators for fractional calculus and related higher transcendental functions. Recent IDE ideas suggest that in some fields, including engineering, physics, medicine and biology, things change their state rapidly in specific locations [10, 39].

Mahmudov et al. [28] studied the BVPs under the fractional integral conditions known as the Katugampola (or, alternatively, the Erd'elyi-Kober type) conditions. Asawasamrit et al. [5] took into mind the nonlocal BVPs for Hilfer FDEs. Da Costa Sousa et al. [36] used the η-Hilfer operator to study a Gronwall inequality. The sequential nonlocal BVPs for inclusions and FIDEs of the Hilfer type were studied by Phuangthong et al. (reference [31]). Wang et al. studied the existence results for FDEs with boundary integral multipoint conditions. In [35], Sitho et al. looked into the BVPs for η-Hilfer for FDEs. Sudsutad et al. examined the stability and existence of the η-Hilfer FIDE in [41]. When it comes to Hilfer IDEs. In [42], Yu studied β-Hyers-Ulam stability for a particular class of FDEs. The investigation of FDEs with delayed impulses was done by Zhang et al. Under impulsive conditions, η-Hilfer FDEs were looked at in the references [41].

The following proportional fractional derivatives were investigated by Abbas [1]:

$$\begin{aligned} {}_{\mathrm{a}}\mathcal{D}^{\mathrm{p,q,g}}\varepsilon(\varpi) &= \Im\left(\varpi, \varepsilon(\varpi), {}_{\mathrm{a}_1}\mathcal{I}^{\mathrm{r,q,g}}\varepsilon(\varpi)\right), \qquad \varpi \in (\mathrm{r}_\Lambda, \varpi_{\Lambda+1}], \\ \varepsilon(\varpi) &= \eta_\Lambda\left(\varpi, \varepsilon\left(\mathrm{t}_\Lambda^+\right)\right), \qquad \varpi \in (\varpi_\Lambda, \mathrm{r}_\Lambda], \qquad \Lambda = 1, \ldots, \hbar, \\ \mathcal{I}^{1-\mathrm{p,q,g}}\varepsilon(\mathrm{a}_1) &= \varepsilon_0 \in \Re, \end{aligned}$$

where ${}_{\mathrm{a}1}\mathcal{D}^{\mathrm{p,q,g}}$ and ${}_{\mathrm{a}_1}\mathcal{I}^{\mathrm{r,q,g}}$. Indicate the proportionate fractional integral and derivative, and note that the function Xi is continuous.

The following form represents the non-local boundary conditions of the fractional Hilfer derivative was explored by Nuchpong et al. in [30]:

$$\begin{aligned} {}^{\mathrm{H}}\mathcal{D}^{\mathrm{p,q}}\varepsilon(\Upsilon) &= \Im\left(\Upsilon, \varepsilon(\Upsilon), \mathcal{I}^{\delta}\varepsilon(\Upsilon)\right), \qquad \Upsilon \in [\mathrm{a}_1, \mathrm{a}_2], \\ \varepsilon(\mathrm{a}_1) = 0, &\qquad \wp + \int_{\mathrm{a}_1}^{\mathrm{a}_2} \varepsilon(l)\mathrm{d}l = \sum_{\Lambda=1}^{\hbar-2} \varsigma_\Lambda \varepsilon(\vartheta_\Lambda), \end{aligned}$$

and $\mathcal{I}^{\delta}$-R-L, ${}^{\mathrm{H}}\mathcal{D}^{\mathrm{p,q}}$ fractional Hilfer derivative and $\Im$ is continuous.

In their study of the implicit non-instantaneous BVP for generalized fractional-order Hilfer derivatives of the following form, Salim et al. in [33]:

$$\begin{aligned} \left({}^{\alpha}\mathcal{D}_{\tau^+}^{\mathrm{p,q}}\varepsilon\right)(\Upsilon) &= \Im\left(\Upsilon, \varepsilon(\Upsilon), \left({}^{\alpha}\mathcal{D}^{\mathrm{p,q}}\varepsilon\right)(\Upsilon)\right), \qquad \Upsilon \in \mp_\Lambda, \\ \varepsilon(\Upsilon) &= \eth_\Lambda \Upsilon, \varepsilon(\Upsilon)), \qquad \Upsilon \in [\Upsilon_\Lambda, \mathrm{r}_\Lambda], \qquad \Lambda = 1, \ldots, \hbar, \\ \varphi_1\left({}^{\alpha}\mathcal{I}_{a_1^+}^{1-\epsilon}\right)(\mathrm{a}_1) &+ \varphi_2\left({}^{\alpha}\mathcal{I}_{\tau^+}^{1-\epsilon}\right)(\mathrm{a}_2) = \varphi_3, \end{aligned}$$

where ${}^{\alpha}\mathcal{I}_{a_1^+}^{1-\epsilon}$ and ${}^{\alpha}\mathcal{D}_{\tau^+}^{\mathrm{p,q}}$ are fractional integral and fractional generalized Hilfer derivative and $\Im$ is continuous.

We offer some uniqueness and existence results for the following fractional issue, which are inspired by the aforementioned papers:

$$ {}^{\mathrm{H}}\mathcal{D}^{\mathrm{p,q;}\eta}\varepsilon(\Upsilon) = \Im(\Upsilon, \varepsilon(\Upsilon), \mathcal{B}\varepsilon(\Upsilon), \mathcal{C}\varepsilon(\Upsilon)), \qquad \Upsilon \in (\mathrm{r}_\Lambda, \Upsilon_{\Lambda+1}], \tag{1}$$

$$\varepsilon(\Upsilon) = \eth_\Lambda(\Upsilon, \varepsilon(\Upsilon)), \qquad \Upsilon \in [\Upsilon_\Lambda, \mathrm{r}_\Lambda], \qquad \Lambda = 1, \ldots, \hbar, \tag{2}$$

$$\varepsilon(0) = 0, \tag{3}$$

$$\varepsilon(\Im) = \sum_{\Lambda=1}^{\hbar} \upsilon_\Lambda \mathcal{I}^{\varsigma^\Lambda}\varepsilon(\upsilon_\Lambda), \qquad \upsilon_\Lambda \in \Re, \qquad \upsilon_\Lambda \in [0, \Im], \tag{4}$$

where $\mathcal{I}^{\varsigma_\Lambda}$-is η R-L, $\mathrm{p} \in (1,2)$, $\mathrm{q} \in [0,1]$, and $0 = \mathrm{r}_0 < \Upsilon_1 \leq \Upsilon_2 < \cdots < \Upsilon_\hbar \leq \mathrm{r}_\hbar \leq \mathrm{r}_{\hbar+1} = \mathrm{r}_\Lambda$, $\Im : [0, \mathrm{r}_\Lambda] \times \Re^3 \longrightarrow \Re$ and $\eth_\Lambda : [\Upsilon_\Lambda, r_\Lambda] \times \Re \longrightarrow \Re$ are continuous. Also,

$$\mathcal{B}\varepsilon(\Upsilon) = \int_0^{\Upsilon} \Lambda(\Upsilon, t)\varepsilon(l)\mathrm{d}t, \qquad \mathcal{C}\varepsilon(\Upsilon) = \int_0^{\mathrm{r}_\Lambda} \Lambda_1(\Upsilon, t)\varepsilon(l)\mathrm{d}t,$$

and $\Lambda, \Lambda_1 \in \mathcal{C}(\mathrm{D}, \Re^+)$, and $\mathrm{D} := \{\Upsilon, \mathrm{r}) \in \Re^2 : 0 \leq \mathrm{r} \leq \Upsilon \leq \mathrm{r}_\Lambda\}$.

2 An Auxiliary Result

Suppose that $\mathrm{PC}([0, \Im], \Re) := \{\varepsilon : [0, \Im] \to \Re : \varepsilon \in \mathcal{C}(j_\Lambda, \Upsilon_{\Lambda+1}), \Re\}$ is continuous. Assume that $\exists\, \varepsilon(\Upsilon_\Lambda^-)$and $\varepsilon(\Upsilon_\Lambda^+)$, where $\varepsilon(\Upsilon_\Lambda^-) = \varepsilon(\Upsilon_\Lambda^+)$ with $\|\varepsilon\|_{\mathrm{PC}} := \sup\{|\varepsilon(\Upsilon)| : 0 \leq j \leq \Im\}$.

Set $\mathbb{PC}([0, \Im], \Re) := \{\varepsilon \in \mathrm{PC}([0, \Im], \Re) : \varepsilon' \in \mathrm{PC}([0, \Im], \Re)\}$ and $\|\varepsilon\|_{\mathbb{PC}^\infty} := \max\{\|\varepsilon\|_{\mathrm{PC}}, \|\varepsilon'\|_{\mathrm{PC}}\}$.

Definition 2.1. [31] *The R-L derivatives and integrals of Υ with η function be given by:*

$$\mathrm{D}^{\mathrm{p};\eta}\Im(\Upsilon) = \left(\frac{1}{\eta'\Upsilon)}\frac{\mathrm{d}}{\mathrm{d}\Upsilon}\right)^{\sigma} \mathcal{I}^{\sigma-\mathrm{p};\eta}\Im\Upsilon) = \frac{1}{\Gamma(\sigma-\mathrm{p})}\left(\frac{1}{\eta'\Upsilon)}\frac{\mathrm{d}}{\mathrm{d}}\right)^{\sigma}\int_0^{\Upsilon} \eta'(l)(\eta\Upsilon) - \eta(l))^{\sigma-\mathrm{p}-1}\Im(l)\mathrm{d}l,$$

and

$$\mathcal{I}^{\mathrm{p};\eta}\Im(\Upsilon) = \frac{1}{\Gamma(\mathrm{p})}\int_0^{\Upsilon} \eta'(l)(\eta\Upsilon) - \eta(l))^{\mathrm{p}-1}\Im(l)\mathrm{d}l,$$

respectively.

Definition 2.2. [18] *The η fractional Hilfer derivative of the function Υ is given by:*

$${}^{\mathrm{H}}\mathcal{D}^{\mathrm{p},\mathrm{q};\eta}\Im(\Upsilon) = \mathcal{I}^{\mathrm{q}(\sigma-\mathrm{p});\eta}\left(\frac{1}{\eta'\Upsilon)}\frac{\mathrm{d}}{\mathrm{d}_\Upsilon}\right)^{\sigma}\mathcal{I}^{(1-\mathrm{q})(\sigma-\mathrm{p});\eta}\Im\Upsilon), \qquad \sigma-1<\mathrm{p}<\sigma, \qquad 0\leq\mathrm{q}\leq 1.$$

Lemma 2.1. [27] *Let $\delta > 0$ and $\mathrm{p}, \iota > 0$. Then,*

(1) $\mathcal{I}^{\mathrm{p};\Omega}\mathcal{I}^{i;\Omega}\varepsilon(\Upsilon) = \mathcal{I}^{\mathrm{p}+\iota;\Omega}\varepsilon(j)$,

(2) $\mathcal{I}^{\mathrm{p};\Omega}(\Omega\Upsilon) - \Omega(0))^{\delta-1} = \dfrac{\Gamma(\delta)}{\Gamma(\mathrm{p}+\delta)}(\Omega\Upsilon) - \Omega(0))^{\mathrm{p}+\delta-1}$.

When ${}^{\mathrm{H}}\mathcal{D}^{\mathrm{p},\mathrm{q};\Omega}(\Omega\Upsilon) - \Omega(0))^{\theta-1} = 0$.

Lemma 2.2. [38] *Assume that $\Im \in \mathcal{L}(\mathrm{a}_1, \mathrm{a}_2), \sigma - 1 < \mathrm{p} \leq \sigma$, and $\mathcal{I}^{(\sigma-\mathrm{p})(1-\mathrm{q})}\mathfrak{y} \in \mathcal{AC}^{\Lambda}[\mathrm{a}_1, \mathrm{a}_2]$, $\sigma \in \mathbb{N}$. Then,*

$$\left(\mathcal{I}^{\mathrm{p};\eta}; \eta^{\mathrm{H}}\mathcal{D}^{\mathrm{p},\mathrm{q};\eta}\Im\right)\Upsilon) = \Im\Upsilon) - \sum_{\Lambda=1}^{\sigma} -\frac{(\eta\Upsilon) - \eta(0)}{\Gamma(\theta-\Lambda+1)}\Im_\eta^{[\sigma-\Lambda]}\lim_{\Upsilon\to\mathrm{a}_1^+}\left(\mathcal{I}^{(\sigma-\mathrm{p})(1-\mathrm{q});\eta}\Im\right)\Upsilon),$$
$$\theta = \mathrm{p} + \sigma\mathrm{q} - \mathrm{p}\mathrm{q},$$

where $\Im_\eta^{[\sigma-\Lambda]} = \left(\dfrac{1}{\eta'\Upsilon)}\dfrac{\mathrm{d}}{\mathrm{d}\Upsilon}\right)^{\sigma-\Lambda}\Im\Upsilon)$, *and*

$$\left|{}^{\mathrm{H}}\mathcal{D}^{\mathrm{p},\mathrm{q};\eta}\varepsilon(\Upsilon) - \Im(d, \varepsilon(d), \chi\Upsilon(d), \phi\varepsilon(d))\right| \leq \epsilon. \tag{5}$$

Lemma 2.3. *Let $\varepsilon \in \mathrm{PC}\left([0, \Im], \Re\right)$ a function given as follows:*

$$\varepsilon(\Upsilon) := \begin{cases} \eth_\Lambda\left(\mathrm{r}_\hbar\right) + \frac{1}{\Gamma(\mathrm{p})}\int_{\mathrm{a}_1}^{\Upsilon} \eta'(l)(\eta\Upsilon) - \eta(l))^{\mathrm{p}-1}\omega(l)\mathrm{d}l \\ + \frac{\left(\eta(j)-\eta(()\right)^{\theta-1}}{\Delta\Gamma(\mathrm{p})}\left[\sum_{\Lambda=1}^{\hbar} \upsilon_\Lambda \int_0^{\upsilon_\Lambda} \eta'\Upsilon)\left(\eta\left(\upsilon_\Lambda\right) - \eta(l)\right)^{\mathrm{p}-1}\omega(l)\mathrm{d}l\right], \\ \qquad \Upsilon \in \left[0, \Upsilon_1\right], \quad \eth_\Lambda(\Upsilon), \Upsilon \in \left[\Upsilon_\Lambda, \mathrm{r}_\Lambda\right], \quad \Lambda = 1, 2, \ldots, \hbar, \\ \\ \eth_\Lambda\left(\mathrm{r}_\Lambda\right) + \frac{1}{\Gamma(\mathrm{p})}\int_0^{\Upsilon} \eta'(l)(\eta\Upsilon) - \eta(l))^{\mathrm{p}-1}\omega(l)\mathrm{d}l \\ - \frac{1}{\Gamma(\mathrm{p})}\int_0^{\mathrm{r}_\Lambda} \eta'(l)\left(\eta\left(\mathrm{r}_\Lambda\right) - \eta(\mathrm{r})\right)^{\mathrm{p}-1}\omega(l)\mathrm{d}l, \\ \qquad \Upsilon \in \left(\mathrm{r}_\Lambda, \Upsilon_{\Lambda+1}\right], \quad \Lambda = 1, 2, \ldots, \hbar. \end{cases} \tag{6}$$

is a solution of the following system:

$$\begin{aligned} {}^{\mathrm{H}}\mathcal{D}^{\mathrm{p},\mathrm{q};\eta}\varepsilon(\Upsilon) &= \omega\Upsilon), \qquad \Upsilon \in \left(\mathrm{r}_\Lambda, \Upsilon_{\Lambda+1}\right] \subset [0, \Im], \qquad 0 < \mathrm{p} < 1, \\ \varepsilon(\Upsilon) &= \eth_\Lambda\Upsilon), \qquad \Upsilon \in \left[\Upsilon_\Lambda, \mathrm{r}_\Lambda\right], \qquad \Lambda = 1, \ldots, \hbar, \\ \varepsilon(0) &= 0, \\ \varepsilon\left(\Im\right) &= \sum_{\Lambda=1}^{\hbar} \upsilon_\Lambda \mathcal{I}^{\varsigma_\Lambda}\varepsilon\left(\upsilon_\Lambda\right), \end{aligned} \tag{7}$$

where,

$$\Delta := (\eta\Upsilon) - \eta(0))^{\theta-1}\sum_{\Lambda=1}^{\hbar} \upsilon_\Lambda\left(\eta\left(\upsilon_\Lambda\right) - \eta(0)\right)^{\theta-1} \neq 0.$$

Proof. Let $\varepsilon(\Upsilon)$ satisfies (7). Integrating (7) for $\Upsilon \in [0, \Upsilon_1]$, we have

$$\varepsilon(\Upsilon) = \varepsilon\left(\Im\right) + \frac{1}{\Gamma(\mathrm{p})}\int_0^{\Upsilon} \eta'(l)(\eta\Upsilon) - \eta(l))^{\mathrm{p}-1}\omega(l)\mathrm{d}t. \tag{8}$$

On other hand, if $j \in \left(\mathrm{r}_\Lambda, \Upsilon_{\Lambda+1}\right], \Lambda = 1, 2, \ldots, \hbar$, when integrating (7), we have

$$\varepsilon(\Upsilon) = \varepsilon\left(l_\Lambda\right) + \frac{1}{\Gamma(\mathrm{p})}\int_{\mathrm{r}_\Lambda}^{j} \eta'(l)(\eta\Upsilon) - \eta(l))^{\mathrm{p}-1}\omega(l)\mathrm{d}l. \tag{9}$$

From $\varepsilon(\Upsilon) = \eth_\Lambda\Upsilon), \Upsilon \in [\Upsilon_\Lambda, \mathrm{r}_\Lambda]$, we get

$$\varepsilon\left(l_\Lambda\right) = \eth_\Lambda\left(\mathrm{r}_\Lambda\right). \tag{10}$$

Consequently, from Eqs. (9) and (10), we get

$$\varepsilon(\Upsilon) = \eth_\Lambda\left(\mathrm{r}_\Lambda\right) + \frac{1}{\Gamma(\mathrm{p})}\int_0^{\Upsilon} \eta'(l)(\eta\Upsilon) - \eta(l))^{\mathrm{p}-1}\omega(l)\mathrm{d}l, \tag{11}$$

and

$$\varepsilon(\Upsilon) = \eth_\Lambda\left(\mathrm{r}_\Lambda\right) + \frac{1}{\Gamma(\mathrm{p})}\int_0^{\Upsilon} \eta'(l)(\eta\Upsilon) - \eta(l))^{\mathrm{p}-1}\omega(l)\mathrm{d}l - \frac{1}{\Gamma(\mathrm{p})}\int_0^{\mathrm{r}_\Lambda} \left(\eta'(l)\eta\left(\mathrm{r}_\Lambda\right) - \eta(\mathrm{r})\right)^{\mathrm{p}-1}\omega(l)\mathrm{d}l. \tag{12}$$

We now show that ε satisfies Eq. (7). Clearly, $\varepsilon(0)=0$,

$$\begin{aligned}\sum_{\Lambda=1}^{\hbar} v_\Lambda \mathcal{I}^{\varphi_\Lambda}\varepsilon(v_\Lambda) &= \sum_{\Lambda=1}^{\hbar} v_\Lambda \frac{(\eta\Upsilon)-\eta(0))^{\mathrm{p}-1}}{\Delta\Gamma(\theta)}\left[\sum_{\Lambda=1}^{\hbar} v_\Lambda \mathcal{I}^{\mathrm{p}+\varphi_\Lambda;\eta}\omega(v_\Lambda)-\mathcal{I}^{\alpha;\eta}\omega(\mathrm{a}_2)\right]\\ &\quad+\sum_{\Lambda=1}^{\hbar} v_\Lambda \mathcal{I}^{\alpha+\varphi_\Lambda}\omega(v_\Lambda)\\ &= \frac{(\eta\Upsilon)-\eta(0))^{\theta-1}}{\Delta}\left[\sum_{\Lambda=1}^{\hbar} v_\Lambda \mathcal{I}^{\mathrm{p}+\varphi_\Lambda;\eta}\omega(v_\Lambda)\right]+\mathcal{I}^{\mathrm{p};\eta}\omega(\Im)\\ &=\varepsilon(\Im).\end{aligned} \tag{13}$$

Now, it's clear that (8), (12) and (13)⇒(6), hence the proof. □

3 Main Results

Theorem 3.1. *Supposes that the premise below is accurate:*

$(Al_1): \exists\ \mathcal{L},\mathcal{G},\mathcal{N}_1,\mathcal{M},\mathcal{M}_1,\mathcal{L}_{\mathrm{h}_\Lambda}>0,$ *for* $\varepsilon_i,\omega_i,u_i,v_i,\vartheta\in\Re,\quad i=1,2,$ *such that:*

$$\begin{aligned}|\Im[\Upsilon,\varepsilon_1,\omega_1,u_1)-\Im(j,\varepsilon_2,\omega_2,u_2)| &\le \mathcal{L}|\varepsilon_1-\varepsilon_2|+\mathcal{G}|\omega_1-\omega_2|+\mathcal{N}_1|u_1-u_2|,\qquad \Upsilon\in[0,\Im],\\ |\Lambda\Upsilon,l,\vartheta)-\Lambda\Upsilon,l,v)| &\le \mathcal{M}|\vartheta-v|,\qquad \textit{for }\Upsilon\in[\Upsilon_\Lambda,\mathrm{r}_\Lambda],\\ |\Lambda_1\Upsilon,l,\vartheta)-\Lambda_2\Upsilon,l,v)| &\le \mathcal{M}_1|\vartheta-v|,\qquad \textit{for }\Upsilon\in[\Upsilon_\Lambda,\mathrm{r}_\Lambda],\\ |\eth_\Lambda(\Upsilon,\mathrm{v}_1)-\eth_\Lambda(\Upsilon,\mathrm{v}_2)| &\le \mathcal{L}_{\mathrm{h}_\Lambda}|\mathrm{v}_1-\mathrm{v}_2|.\end{aligned}$$

If,

$$\begin{aligned}\mathcal{Z}:=\max\Bigg\{&\max_{\Lambda=1,2,\ldots,\hbar}\mathcal{L}_{\mathrm{h}_\Lambda}+\frac{(\mathcal{L}+\mathcal{G}\mathcal{M}+\mathcal{N}_1\mathcal{M}_1)}{\Gamma(\mathrm{p}+1)}\left(\Upsilon^{\mathrm{p}}_{\Lambda+1}+\mathrm{r}^{\mathrm{p}}_\Lambda\right),\\ &\mathcal{L}_{\mathrm{h}_\Lambda}+(\mathcal{L}+\mathcal{G}\mathcal{M}++\mathcal{N}_1\mathcal{M}_1)\\ &\times\left\{\frac{(\eta\Upsilon)-\eta(0))^{\theta-1}}{|\Delta|\Gamma(\theta)}\left[\sum_{\Lambda=1}^{\hbar}|v_\Lambda|\frac{(\eta(v_\Lambda)-\eta(0))^{\mathrm{p}+\varphi_\Lambda;\eta}}{\Gamma(\mathrm{p}+\varphi_\Lambda+1)}\right]+\frac{(\eta\Upsilon)-\eta(0))^{\mathrm{p}}}{\Gamma(\mathrm{p}+1)}\right\}\Bigg\}<1,\end{aligned} \tag{14}$$

then the problem given by (1)-(4) *has a unique solution on* $[0,\Im]$.

Proof. Let expand $\mathcal{N}:\mathrm{PC}([0,\Im],\Re)\longrightarrow\mathrm{PC}([0,\Im],\Re)$ by

$$(\mathcal{N}\varepsilon)\Upsilon:=\begin{cases}\eth_\hbar(\mathrm{r}_\hbar,\varepsilon(l_\hbar))+\dfrac{1}{\Gamma(\mathrm{p})}\displaystyle\int_0^\Upsilon\eta'(l)(\eta\Upsilon)-\eta(l))^{\mathrm{p}-1}\Im\big(l,\varepsilon(l),\mathcal{B}\varepsilon(l),\mathcal{C}\varepsilon(l)\big)\mathrm{d}l\\ +\dfrac{\big(\eta(j)-\eta(0)\big)^{\theta-1}}{\Delta}\left[\displaystyle\sum_{\Lambda=1}^{\hbar}v_\Lambda\int_0^{v_\Lambda}\eta'(j)\big(\eta(v_\Lambda)-\eta(l)\big)^{\mathrm{p}-1}\Im\big(v_\Lambda,\varepsilon(v_\Lambda),\mathcal{B}\varepsilon(v_\Lambda),\mathcal{C}\varepsilon(v_\Lambda)\big)\right],\\ \qquad\Upsilon\in[0,\Upsilon_1],\quad \eth_\Lambda(\Upsilon),\quad \Upsilon\in[\Upsilon_\Lambda,\mathrm{r}_\Lambda],\quad \Lambda=1,2,\ldots,\hbar,\\ \eth_\Lambda(\mathrm{r}_\Lambda)+\dfrac{1}{\Gamma(\mathrm{p})}\displaystyle\int_0^\Upsilon\eta'(l)(\eta\Upsilon)-\eta(l))^{\mathrm{p}-1}\Im(l,\varepsilon(l),\mathcal{B}\varepsilon(l),\mathcal{C}\varepsilon(l))\mathrm{d}l\\ -\frac{1}{\Gamma(\mathrm{p})}\int_0^{\mathrm{r}_\Lambda}\eta'(l)\left(\eta(\mathrm{r}_\Lambda)-\eta(\mathrm{r})\right)^{\mathrm{p}-1}\Im(l,\varepsilon(l),\mathcal{B}\varepsilon(l),\mathcal{C}\varepsilon(l))\mathrm{d}l,\\ \qquad\Upsilon\in(\mathrm{r}_\Lambda,\Upsilon_{\Lambda+1}],\quad\Lambda=1,2,\ldots,\hbar.\end{cases}$$

Thus, $\mathcal{N}$ is well defined,

Case 1. $\mathcal{N}$ is a contraction.
Let $\varepsilon, \bar{\varepsilon} \in \mathrm{PC}([0, \Im], \Re)$ and $\Upsilon \in [0, \Upsilon_1]$, we get

$$|(\mathcal{N}\varepsilon)\Upsilon) - (\mathcal{N}\varepsilon)\Upsilon)|$$
$$\leq \mathcal{L}_{\mathrm{h}_\Lambda} + (\mathcal{L} + \mathcal{G}\mathcal{M} + \mathcal{N}_1\mathcal{M}_1)$$
$$\left\{ \frac{(\eta\Upsilon) - \eta(0))^{\theta-1}}{|\Delta|\Gamma(\theta)} \left[\sum_{\Lambda=1}^{\hbar} |v_\Lambda| \frac{(\eta(v_\Lambda) - \eta(0))^{\mathrm{p}+\varphi_\Lambda;\eta}}{\Gamma(\mathrm{p} + \varphi_\Lambda + 1)} \right] + \frac{(\eta\Upsilon) - \eta(0))^{\mathrm{p}}}{\Gamma(\mathrm{p}+1)} \right\} \|\varepsilon - \varepsilon\|_{\mathrm{PC}}.$$

Case 2. Let $j \in [\Upsilon_\Lambda, r_\Lambda]$, we have

$$|(\mathcal{N}\varepsilon)\Upsilon) - (\mathcal{N}\bar{\varepsilon})\Upsilon)| \leq |\eth_\Lambda\Upsilon, \varepsilon(\Upsilon)) - \eth_\Lambda\Upsilon, \bar{\varepsilon}\Upsilon))|$$
$$\leq \mathcal{L}_{\mathrm{h}_\Lambda}\|\varepsilon - \bar{\varepsilon}\|_{\mathrm{PC}}.$$

Case 3. Letting $j \in (\mathrm{r}_\Lambda, \Upsilon_{\Lambda+1}]$, we obtain

$$|(\mathcal{N}\varepsilon)\Upsilon) - (\mathcal{N}\bar{\varepsilon})\Upsilon)|$$
$$\leq| \eth_\Lambda(\mathrm{r}_\Lambda, \varepsilon(l_\Lambda)) - \eth_\Lambda(l_\Lambda, \bar{\varepsilon}(\mathrm{r}_\Lambda))$$
$$+ \frac{1}{\Gamma(\mathrm{p})} \int_0^{\jmath} \Upsilon - \imath)^{\mathrm{p}-1} |\Im(l, \varepsilon(l), \mathcal{B}\varepsilon(l), \mathcal{C}\varepsilon(l)) - \Im(l, \bar{\varepsilon}(l), \mathcal{B}\bar{\varepsilon}(l), \mathcal{C}\bar{\varepsilon}(l))|\, \mathrm{d}l$$
$$+ \frac{1}{\Gamma(\mathrm{p})} \int_0^{\mathrm{r}_\Lambda} (\mathrm{r}_\Lambda - \mathrm{r})^{\mathrm{p}-1} |\Im(l, \varepsilon(l), \mathcal{B}\varepsilon(l), \mathcal{C}\varepsilon(l)) - \Im(l, \bar{\varepsilon}(l), \mathcal{B}\bar{\varepsilon}(l), \mathcal{C}\bar{\varepsilon}(l))|\mathrm{d}l$$
$$\leq \left[\mathcal{L}_{\mathrm{h}_\Lambda} + \frac{(\mathcal{L} + \mathcal{G}\mathcal{M} + \mathcal{N}_1\mathcal{M}_1)}{\Gamma(\mathrm{p}+1)} \left(\Upsilon_{\Lambda+1}^{\mathrm{p}} + \mathrm{r}_\Lambda^{\mathrm{p}} \right) \right] \|\varepsilon - \bar{\varepsilon}\|_{\mathrm{PC}}.$$

Thus, $\mathcal{N}$ be contraction when,

$$\mathcal{Z} = \left[\mathcal{L}_{\mathrm{h}_\Lambda} + \frac{(\mathcal{L} + \mathcal{G}\mathcal{M} + \mathcal{N}_1\mathcal{M}_1)}{\Gamma(\mathrm{p}+1)} \left(j_{\Lambda+1}^{\mathrm{p}} + \mathrm{r}_\Lambda^{\mathrm{p}} \right) \right] < 1.$$

Then, the Eqs (1)-(4) has a unique solution when $\varepsilon \in \mathrm{PC}\big([0, \Im], \Re\big)$. □

Theorem 3.2. *Supposes that* (AI_1) *is hold and the following assumption satisfied:*

$(A1_2)$: $\exists\, \mathcal{L}_{\mathrm{g}_\Lambda} > 0$ *and*

$$|\Im(j,\ \mathrm{W}_1, \omega_1, u_1)| \leq \mathcal{L}_{\mathrm{g}_\Lambda}(1 + |\mathrm{W}_1| + |\omega_1| + |u_1|), \quad j \in [\mathrm{r}_\Lambda, \Upsilon_{\Lambda+1}], \forall\, \mathrm{W}_1, \omega_1, u_1 \in \Re.$$

(A_3) : $\exists$ *A function* $\kappa_\Lambda\Upsilon), \Lambda = 1, 2, \ldots, \hbar$, *and*

$$|\eth_\Lambda(d,\ \mathrm{s}_1, \omega_1)| \leq \kappa_\Lambda(d), \quad d \in [j_\Lambda, \mathrm{r}_\Lambda], \forall \mathrm{s}_1, \omega_1 \in \Re.$$

Let $\mathcal{M}_\Lambda := \sup_{\Upsilon\in[\Upsilon_\Lambda, r_r]} K_\Lambda\Upsilon) < \infty$ *and* $\mathcal{K} := \max \mathcal{L}_{\mathrm{g}_\Lambda} < 1, \forall\ \Lambda = 1, 2, \ldots, \hbar$.
Then Eqs. (1)-(4) *have at least one solution on* $[0, \Im]$.

Proof. Supposes that,

$$\mathcal{B}_{\mathrm{p,r}} := \{\varepsilon \in \mathrm{PC}([0, \Im], \Re) : \|\varepsilon\|_{\mathrm{PC}} \leq \mathrm{r}\}.$$

Let Q and ∇ are factors on $\mathcal{B}_{\mathrm{p,r}}$ given by:

$$Q\varepsilon(\Upsilon) := \begin{cases} \eth_{\hbar}\left(\mathrm{r}_{\hbar}, \varepsilon\left(l_{\hbar}\right)\right), & \Upsilon \in \left[0, \Upsilon_1\right], \\ \eth_{\Lambda}(j, \varepsilon(\Upsilon)), & \Upsilon \in \left[\Upsilon_{\Lambda}, \mathrm{r}_{\Lambda}\right], \quad \Lambda = 1, 2, \ldots, \hbar, \\ \eth_{\Lambda}\left(\mathrm{r}_{\Lambda}, \varepsilon\left(l_{\Lambda}\right)\right), & \Upsilon \in \left(\mathrm{r}_{\Lambda}, \Upsilon_{\Lambda+1}\right], \quad \Lambda = 1, 2, \ldots, \hbar, \end{cases}$$

and

$$\nabla\varepsilon(\Upsilon) := \begin{cases} \dfrac{1}{\Gamma(\mathrm{p})} \displaystyle\int_{\mathrm{a}_1}^{\Upsilon} \eta'(l)(\eta\Upsilon) - \eta(l))^{\mathrm{p}-1} \Im(l, \varepsilon(l), \mathcal{B}\varepsilon(l), \mathcal{C}\varepsilon(l)) \mathrm{d}l \\ + \dfrac{(\eta\Upsilon) - \eta(0))^{\theta-1}}{\Delta} \left[\displaystyle\sum_{\Lambda=1}^{\hbar} v_{\Lambda} \int_0^{v_{\Lambda}} \eta'\Upsilon)\left(\eta\left(v_{\Lambda}\right) - \eta(l)\right)^{\mathrm{p}-1} \Im\left(v_{\Lambda}, \varepsilon\left(v_{\Lambda}\right), \mathcal{B}\varepsilon\left(v_{\Lambda}\right), \mathcal{C}\varepsilon\left(v_{\Lambda}\right)\right)\right], \\ \qquad \Upsilon \in \left[0, \Upsilon_1\right], 0, \Upsilon \in \left[\Upsilon\Lambda, \mathrm{r}_{\Lambda}\right], \quad \Lambda = 1, 2, \ldots, \hbar, \\ \dfrac{1}{\Gamma(\mathrm{p})} \displaystyle\int_0^{\Upsilon} \eta'(l)(\eta\Upsilon) - \eta(l))^{\mathrm{p}-1} \Im(l, \varepsilon(l), \mathcal{B}\varepsilon(l), \mathcal{C}\varepsilon(l)) \mathrm{d}l \\ - \dfrac{1}{\Gamma(\mathrm{p})} \displaystyle\int_0^{r_{\Lambda}} \eta'(l)\left(\eta\left(\mathrm{r}_{\Lambda}\right) - \eta(l)\right)^{\mathrm{p}-1} \Im(l, \varepsilon(l), \mathcal{B}\varepsilon(l), \mathcal{C}\varepsilon(l)) \mathrm{d}l, \\ \qquad \Upsilon \in \left(\mathrm{r}_{\Lambda}, \Upsilon_{\Lambda+1}\right], \quad \Lambda = 1, 2, \ldots, \hbar. \end{cases}$$

Step 1: Let $\varepsilon \in \mathcal{B}_{\mathrm{p,r}}$, we get $Q\varepsilon + \nabla\varepsilon \in \mathcal{B}_{\mathrm{p,r}}$.

Case 1: For $j \in [0, \Upsilon_1]$, we get

$$\begin{aligned} &|Q\varepsilon + \nabla\bar{\varepsilon}| \\ &\leq \left|\eth_{\hbar}\left(\mathrm{r}_{\hbar}, \varepsilon\left(l_{\hbar}\right)\right)\right| + \frac{1}{\Gamma(\mathrm{p})} \int_0^{\Upsilon} \Upsilon - l)^{\mathrm{p}-1} |\Im(l, \varepsilon(l), \mathcal{B}\varepsilon(l), \mathcal{C}\varepsilon(l))| \mathrm{d}l + \frac{(\eta\Upsilon) - \eta(0))^{\theta-1}}{\Delta} \\ &\quad \times \left[\sum_{\Lambda=1}^{\hbar} v_{\Lambda} \int_0^{v_{\Lambda}} \eta'\Upsilon)\left(\eta\left(v_{\Lambda}\right) - \eta(l)\right)^{\mathrm{p}-1} \Im\left(v_{\Lambda}, \varepsilon\left(v_{\Lambda}\right), \mathcal{B}\varepsilon\left(v_{\Lambda}\right), \mathcal{C}\varepsilon\left(v_{\Lambda}\right)\right) \mathrm{d}v_{\Lambda}\right], \\ &\leq \left[\mathcal{L}_{\mathrm{h}_{\Lambda}} + \left(\mathcal{L} + \mathcal{G}\mathcal{M} + \mathcal{N}_1\mathcal{M}_1\right)\left\{\frac{(\eta\Upsilon) - \eta(0))^{\mathrm{p}}}{\Gamma(\mathrm{p}+1)} + \frac{(\eta\Upsilon) - \eta(0))^{\theta-1}}{|\Delta|\Gamma(\theta)}\right.\right. \\ &\quad \left.\left. \times \left[\sum_{\Lambda=1}^{\hbar} \left|v_{\Lambda}\right| \frac{\left(\eta\left(v_{\Lambda}\right) - \eta(0)\right)^{\mathrm{p}+\varphi_{\Lambda};\eta}}{\Gamma\left(\mathrm{p}+\varphi_{\Lambda}+1\right)}\right]\right\}\right](1+\mathrm{r}) \leq \mathrm{r}. \end{aligned}$$

Case 2: $\forall\ j \in [\Upsilon_{\Lambda}, r_{\Lambda}]$, we get

$$|Q\varepsilon + \nabla\bar{\varepsilon}| \leq \left|\eth_{\Lambda}\left(j,\ \mathrm{W}_1(j)\right)\right| \leq \mathcal{M}_{\Lambda}.$$

Case 3: $\forall\ j \in (r_{\Lambda}, \Upsilon_{\Lambda+1}]$,

$$\begin{aligned} |Q\varepsilon + \nabla\bar{\varepsilon}\Upsilon)| &\leq \left|\eth_{\Lambda}\left(\mathrm{r}_{\Lambda}, \varepsilon\left(l_{\Lambda}\right)\right)\right| + \frac{1}{\Gamma(\mathrm{p})} \int_0^{j} \Upsilon - l)^{\mathrm{p}-1} |\Im(l, \varepsilon(l), \mathcal{B}\varepsilon(l), \mathcal{C}\varepsilon(l))| \mathrm{d}l \\ &\quad + \frac{1}{\Gamma(\mathrm{p})} \int_0^{\mathrm{r}_{\Lambda}} \left(\mathrm{r}_{\Lambda} - \mathrm{r}\right)^{\mathrm{p}-1} |\Im(l, \varepsilon(l), \mathcal{B}\varepsilon(l), \mathcal{C}\varepsilon(l))| \mathrm{d}l, \\ &\leq \mathcal{M}_{\Lambda} + \left[\frac{\mathcal{L}_{\mathrm{g}_{\Lambda}}\left(\mathrm{r}_{\Lambda}^{\mathrm{p}} + j_{\Lambda+1}^{\mathrm{p}}\right)}{\Gamma(\mathrm{p}+1)}\right](1+\mathrm{r}) \leq \mathrm{r}. \end{aligned}$$

Thus,

$$Q\varepsilon + \nabla\varepsilon \in \mathcal{B}_{\mathrm{p,r}}$$

Step 2: Q be contraction on $\mathcal{B}_{\mathrm{p,r}}$.

Case 1: Let $\varepsilon_1, \varepsilon_2 \in \mathcal{B}_{\mathrm{p,r}}$. Hence, when $\Upsilon \in [0, \Upsilon_1]$, we get

$$|Q\varepsilon_1\Upsilon) - Q\varepsilon_2(j)| \leq \mathcal{L}_{\mathrm{g}_\hbar} |\varepsilon_1(\mathrm{r}_\hbar) - \varepsilon_2(\mathrm{r}_\hbar)| \leq \mathcal{L}_{\mathrm{g}_\hbar} \|\varepsilon_1 - \varepsilon_2\|_{\mathrm{PC}}.$$

Case 2: $\forall\ j \in [\Upsilon_\Lambda, \mathrm{r}_\Lambda], \Lambda = 1, 2, \ldots, \hbar$, we have

$$|Q\varepsilon_1\Upsilon) - Q\varepsilon_2\Upsilon)| \leq \mathcal{L}_{\mathrm{g}_\Lambda} \|\varepsilon_1 - \varepsilon_2\|_{\mathrm{PC}}.$$

Case 3: $\forall\ j \in (\mathrm{r}_\Lambda, \Upsilon_{\Lambda+1}]$, we obtain

$$|Q\varepsilon_1\Upsilon) - Q\varepsilon_2\Upsilon)| \leq \mathcal{L}_{\mathrm{g}_\Lambda} \|\varepsilon_1 - \varepsilon_2\|_{\mathrm{PC}}.$$

Then,

$$|Q\varepsilon_1\Upsilon) - Q\varepsilon_2\Upsilon)| \leq \mathcal{K} \|\varepsilon_1 - \varepsilon_2\|_{\mathrm{PC}}.$$

Then, Q be a contraction.

Step 3: ∇ is continuous.

For ε_σ is a sequence in $\mathrm{PC}([0, \Im], \Re)$.

Case 1: $\forall\ \Upsilon \in [0, \Upsilon_1]$, we get

$$|Q\varepsilon_\sigma\Upsilon) - Q\varepsilon(\Upsilon)| \leq \left[\frac{(\eta\Upsilon) - \eta(0))^{\theta-1}}{|\Delta|\Gamma(\theta)}\left[\sum_{\Lambda=1}^{\hbar} |v_\Lambda| \frac{(\eta(v_\Lambda) - \eta(0))^{\mathrm{p}+\varphi_\Lambda;\eta}}{\Gamma(\mathrm{p}+\varphi_\Lambda+1)}\right] + \frac{(\eta\Upsilon) - \eta(0))^{\mathrm{p}}}{\Gamma(\mathrm{p}+1)}\right] \times \|\Im(., \varepsilon_\sigma(.), ., .) - \Im(., \varepsilon(.), ., .)\|_{\mathrm{PC}}.$$

Case 2: $\forall\, j \in [\Upsilon_\Lambda, r_\Lambda]$, we get

$$|Q\varepsilon_\sigma\Upsilon) - Q\varepsilon(j)| = 0.$$

Case 3: $\forall\, \Upsilon \in (\mathrm{r}_\Lambda, \Upsilon_{\Lambda+1}], \Lambda = 1, 2, \ldots, \hbar$, we obtain

$$|Q\varepsilon_\sigma\Upsilon) - Q\varepsilon(\Upsilon)| \leq \frac{(\Upsilon_{\Lambda+1} - \mathrm{r}_\Lambda)}{\Gamma(\mathrm{p}+1)} \|\Im(., \varepsilon_\sigma(.), ., .) - \Im(., \varepsilon(.), ., .)\|_{\mathrm{PC}}.$$

Thus, based on the aforementioned situations, we may say $\|Q\varepsilon_\sigma(j) - Q\varepsilon(j)\|_{\mathrm{PC}} \longrightarrow 0$ as $\sigma \to \infty$.

Step 4: Let's finish by demonstrating Q's compactness.

First of all, Q is constrained uniformly on $\mathcal{B}_{\mathrm{p,r}}$.

Since $\|Q\varepsilon\| \leq \dfrac{\mathcal{L}_{g_\Lambda}(\mathcal{T})}{\Gamma(1+\mathrm{p})} < \mathrm{r}$, therefore, we have Q is constrained uniformly on $\mathcal{B}_{\mathrm{p,r}}$.

We demonstrate how Q converts a bounded set to a $\mathcal{B}_{\mathrm{p,r}}$ equicontinuous set.

Case 1: For $\Upsilon \in [0, \Upsilon_1]$, $0 \leq \mathcal{E}_1 \leq \mathcal{E}_2 \leq \Upsilon_1$, $\varepsilon \in \mathcal{B}_{\mathrm{r}}$, we have

$$|Q\mathcal{E}_2 - Q\mathcal{E}_1| \leq \frac{\mathcal{L}_{\mathrm{g}_\Lambda}(1+\mathrm{r})}{\Gamma(\mathrm{p}+1)} (\mathcal{E}_2 - \mathcal{E}_1).$$

Case 2: $\forall\, j \in [\Upsilon_\Lambda, \mathrm{r}_\Lambda], \Upsilon_\Lambda < \mathcal{E}_1 < \mathcal{E}_2 \leq \mathrm{r}_\Lambda, \varepsilon \in \mathcal{B}_{\mathrm{p,r}}$, we have

$$|Q\mathcal{E}_2 - Q\mathcal{E}_1| = 0.$$

Case 3: $\forall\ \Upsilon \in (\mathrm{r}_\Lambda, \Upsilon_{\Lambda+1}], \mathrm{r}_\Lambda < \mathcal{E}_1 < \mathcal{E}_2 \leq \Upsilon_{\Lambda+1}, \varepsilon \in \mathcal{B}_{\mathrm{p,r}}$, we have

$$|Q\mathcal{E}_2 - Q\mathcal{E}_1| \leq \frac{\mathcal{L}_{\mathrm{g}_\Lambda}(1+\mathrm{r})}{\Gamma(\mathrm{p}+1)}(\mathcal{E}_2 - \mathcal{E}_1).$$

From the above cases, we deduce that $|Q\mathcal{E}_2 - Q\mathcal{E}_1| \longrightarrow 0$ as $\mathcal{E}_2 \longrightarrow \mathcal{E}_1$ and Q is equicontinuous. By applying the Ascoli-Arzela theorem, it can be shown that Q is compact and that Q is relatively compact when $Q(\mathcal{B}_{\mathrm{p,r}})$. As a result, there is at least one fixed point on $[0, T_\star]$ in the equations from (1) to (4). □

4 Examples

Example 4.1. *As an illustration of our problem (1)-(4), consider the boundary value impulsive problem given below:*

$$D^{\mathrm{p,q};\eta}\varepsilon(\Upsilon) = \frac{e^{-\Upsilon}|\varepsilon(\Upsilon)|}{9+e^{-\Upsilon}(1+|\varepsilon(\Upsilon)|)} + \frac{1}{3}\int_0^\Upsilon \mathrm{e}^{-(l-\Upsilon)}\varepsilon(l)\mathrm{d}l + \frac{1}{2}\int_0^1 \mathrm{e}^{\Upsilon-l)}\varepsilon(l)\mathrm{d}l, \qquad \Upsilon \in (0,1], \quad (15)$$

$$\varepsilon(\Upsilon) = \frac{|\varepsilon(\Upsilon)|}{2(1+|\varepsilon(\Upsilon)|)}, \qquad \Upsilon \in \left(\frac{1}{2}, 1\right], \quad (16)$$

$$\varepsilon(0) = 0, \quad (17)$$

$$\varepsilon(1) = \frac{1}{2}\mathcal{I}^{\frac{2}{3}}\varepsilon\left(\frac{7}{5}\right) + \frac{2}{3}\mathcal{I}^{\frac{4}{5}}\varepsilon\left(\frac{9}{5}\right) + \frac{5}{2}\mathcal{I}^{\frac{3}{4}}\varepsilon\left(\frac{7}{2}\right), \quad (18)$$

together with $\mathcal{L} = \mathcal{G} = \mathcal{N}_1 = \frac{1}{10}$, $\mathcal{M} = \frac{1}{3}$, $\mathcal{M}_1 = \frac{1}{2}$, $\mathrm{p} = \frac{5}{7}$, $\theta = \frac{2}{5}$, $\mathcal{L}_{\mathrm{h}_1} = \frac{1}{3}$, $\upsilon_1 = \frac{1}{2}$, $\upsilon_2 = \frac{2}{3}$, $\upsilon_3 = \frac{2}{5}$, $\upsilon_1 = \frac{2}{7}$, $\upsilon_2 = \frac{5}{9}$, $\upsilon_3 = \frac{1}{7}$, $\varphi_1 = \frac{2}{3}$, $\varphi_2 = \frac{4}{5}$, $\varphi_3 = \frac{3}{4}$. *We will examine the expression (14) for the value* $\mathrm{p} \in (1,2)$. *Theorem 3.1 allows us to conclude that:*

$$\mathcal{L}_{\mathrm{h}_\Lambda} + \frac{(\mathcal{L}+\mathcal{G}\mathcal{M}+\mathcal{N}_1\mathcal{M}_1)}{\Gamma(\mathrm{p}+1)}\left(j_{\Lambda+1}^{\mathrm{p}} + \mathrm{r}_\Lambda^{\mathrm{p}}\right) \approx 0.5 < 1,$$

and

$$\mathcal{L}_{\mathrm{h}_\Lambda} + (\mathcal{L}+\mathcal{G}\mathcal{M}+\mathcal{N}_1\mathcal{M}_1)\left\{\frac{(\eta\Upsilon)-\eta(0))^{\mathrm{p}}}{\Gamma(\mathrm{p}+1)} + \frac{(\eta\Upsilon)-\eta(0))^{\theta-1}}{|\Delta|\Gamma(\theta)}\left[\sum_{\Lambda=1}^{\hbar}|\upsilon_\Lambda|\frac{(\eta(\upsilon_\Lambda)-\eta(0))^{\mathrm{p}+\varphi_\Lambda;\eta}}{\Gamma(\mathrm{p}+\varphi_\Lambda+1)}\right]\right\}$$
$$\approx 0.59 < 1.$$

Hence, from Theorem 3.1 the system (15)-(18) has a unique solution on $[0,1]$.

Example 4.2. *Consider the boundary value problem given below:*

$$D^{\frac{6}{5},\frac{1}{6};\eta}\varepsilon(\Upsilon) = \Im\left(\Upsilon, \varepsilon(\Upsilon), I^{\frac{5}{2}}\varepsilon(\Upsilon)\right), \qquad \Upsilon \in \left[\frac{1}{5}, \frac{16}{5}\right], \quad (19)$$

$$\varepsilon\left(\frac{1}{5}\right) = 0, \qquad \int_{\frac{1}{5}}^{\frac{16}{5}} \varepsilon(l)\mathrm{d}l + \frac{4}{5} = \frac{2}{7}\varepsilon\left(\frac{8}{5}\right) + \frac{3}{7}\varepsilon\left(\frac{11}{5}\right) + \frac{4}{7}\varepsilon\left(\frac{13}{5}\right) + \frac{5}{7}\varepsilon\left(\frac{14}{5}\right), \quad (20)$$

where

$$\begin{aligned}&\Im\left(\Upsilon,\varepsilon(\Upsilon),I^{\frac{5}{2}}\varepsilon(\Upsilon)\right)\\&=\left[\frac{5\left(\left(1+\tan^{-1}|\varepsilon(\Upsilon)|+I^{\frac{5}{2}}\varepsilon(\Upsilon),\right)\right)}{8(\Upsilon+100)},\frac{5}{5\Upsilon+749}\left(1+\sin|\varepsilon(\Upsilon)|+\frac{\left|I^{\frac{5}{2}}\varepsilon(\Upsilon)\right|}{1+\left|I^{\frac{5}{2}}\varepsilon(\Upsilon)\right|}\right)\right].\end{aligned}$$

Next we can find that

$$\mathcal{L}_{\mathrm{h}_\Lambda}+\frac{(\mathcal{L}+\mathcal{G}\mathcal{M}+\mathcal{N}_1\mathcal{M}_1)}{\Gamma(\mathrm{p}+1)}\left(j_{\Lambda+1}^{\mathrm{p}}+\mathrm{r}_\Lambda^{\mathrm{p}}\right)\approx 0.63821<1,$$

and

$$\begin{aligned}\mathcal{L}_{\mathrm{h}_\Lambda}+(\mathcal{L}+\mathcal{G}\mathcal{M}+\mathcal{N}_1\mathcal{M}_1)\left\{\frac{(\eta\Upsilon)-\eta(0))^{\mathrm{p}}}{\Gamma(\mathrm{p}+1)}+\frac{(\eta\Upsilon)-\eta(0))^{\theta-1}}{|\Delta|\Gamma(\theta)}\left[\sum_{\Lambda=1}^{\hbar}|v_\Lambda|\frac{(\eta\left(v_\Lambda\right)-\eta(0))^{\mathrm{p}+\varphi_\Lambda;\eta}}{\Gamma\left(\mathrm{p}+\varphi_\Lambda+1\right)}\right]\right\}\\\approx 0.7242<1.\end{aligned}$$

Hence, from Theorem 3.1 the system (19)-(20) *has a unique solution on* $\left[\frac{1}{5},\frac{16}{5}\right]$.

5 Conclusion

With boundary multi-point non-instantaneous conditions and impulsive η-Hilfer fractional V-FIDEs, we have looked into the uniqueness and existence criteria for solutions of a BVP. The fixed-point technique, specifically the Krasnoselskii's fixed-point theorem and the Banach contraction principle, was applied to the presented situation to provide the desired results. Furthermore, we established that the approach is bounded. Our findings in the specified configuration are innovative and significantly advance the body of knowledge in this emerging field of research. Due to the dearth of papers on differential implicit hybrid equations, particularly with impulses non-instantaneous, we believe there are numerous prospective study areas such as linked systems, issues with infinite delays, and many more. All the obtained results are supported by an applicable example to apply and validate them. Therefore, this research study sheds the light on this interesting topic of research and motivates all other researchers to work on further investigation of η-Hilfer non-periodic boundary value problem defined in other fractional derivatives. Also, for future work as a continuation of the study, we aim to analyze model (1) in the generalized Caputo fractional derivatives.

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